

EQUIVARIANT STRONGLY PROJECTIVELY FLAT MAPS OF COMPACT HOMOGENEOUS KÄHLER MANIFOLDS

ISAMI KOGA

ABSTRACT. In [4], the author define projectively flat maps of a compact Kähler manifold into complex Grassmannian manifold. In this article, by focusing on the essence of the result in [4] we define *strongly projectively flat maps* and study such maps. Finally we prove a rigidity of equivariant strongly projectively flat maps of simply connected homogeneous Kähler manifold.

1. INTRODUCTION

Holomorphic maps into the complex projective space have been studied for a long time. In [1] E. Calabi have proved that holomorphic isometric immersions of Kähler manifolds into the complex projective space are rigid and equivariant with respect to the group of automorphisms of the domain. In [7] M. Takeuchi has constructed all holomorphic isometric immersions of homogeneous Kähler manifolds into the complex projective space and has classified the holomorphic isometric immersions of Hermitian symmetric spaces.

In this article, we study holomorphic maps of a compact homogeneous Kähler manifold into the complex Grassmannian manifold.

Let \mathbb{C}^n be an n -dimensional complex vector space with a Hermitian inner product $(\cdot, \cdot)_n$ and $Gr_p(\mathbb{C}^n)$ the complex Grassmannian manifold of complex p -planes in \mathbb{C}^n with the Hermitian metric of Fubini-Study type induced by $(\cdot, \cdot)_n$. Let M be a compact Kähler manifold and $f : M \rightarrow Gr_p(\mathbb{C}^n)$ be a holomorphic map. Then we have a holomorphic vector bundle $f^*Q \rightarrow M$ over M which is the pull-back bundle of the universal quotient bundle $Q \rightarrow Gr_p(\mathbb{C}^n)$ over M by f .

Definition 1 (cf.[4]).

- (1) A holomorphic map $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is called *projectively flat* if the pull-back bundle $f^*Q \rightarrow M$ is projectively flat.
- (2) A holomorphic map $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is called *strongly projectively flat* if there exists a holomorphic Hermitian line bundle $L \rightarrow M$ such that $f^*Q \rightarrow M$ is isomorphic to $\tilde{L} \rightarrow M$ with holomorphic structures and fiber metrics, where $\tilde{L} \rightarrow M$ is orthogonal direct sum of q -copies of $L \rightarrow M$.

Strongly projectively flat condition is a kind of simple extension of a map into the complex projective space with Fubini-Study metric. (For a detail, see the section 3).

Projectively flat condition is defined by the author in [2]. A strongly projectively flat map is projectively flat since the orthogonal direct sum of q -copies of a holomorphic line bundle is projectively flat. In general, the inverse of this asseertion is not true. However, we can show the following assertion.

Proposition 1. *Assume that a holomorphic map $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is an isometric. Then f is strongly projectively flat if and only if f is projectively flat.*

Proof. For a detail, see [4]. Let M be a compact Kähler manifold and $f : M \rightarrow Gr_p(\mathbb{C}^n)$ be a holomorphic isometric projectively flat immersion. It follows from the result in [4] that the curvature R^{f^*Q} of pull-back bundle $f^*Q \rightarrow M$ is expressed as

$$R^{f^*Q} = -\frac{\sqrt{-1}}{q}\omega_M \text{Id}_{Q_x},$$

where q is the rank of $Q \rightarrow M$, ω_M is the Kähler form on M and Id_{Q_x} is the identity map of the fiber of $f^*Q \rightarrow M$ at $x \in M$. Since ω_M is parallel, it follows from the holonomy theorem that there exists a Hermitian line bundle $L \rightarrow M$ such that $f^*Q \rightarrow M$ is isomorphic to the orthogonal direct sum of q -copies of $L \rightarrow M$ as a Hermitian vector bundle.

Therefore $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is strongly projectively flat. \square

In the present paper, our goal is to show the following theorem.

Theorem 1. *Let $M = G/K$ be a compact simply connected Kähler manifold such that G is the isometry group of M and K an isotropy subgroup of G . Let $f : M \rightarrow Gr_p(\mathbb{C}^n)$ be a full holomorphic strongly projectively flat map into the complex Grassmannian manifold. If f is G -equivariant, then there exists an N -dimensional complex vector space W and a holomorphic map $f_0 : M \rightarrow Gr_{N-1}(W)$ such that \mathbb{C}^n is regarded as the orthogonal direct sum of q -copies of W and f is congruent to the following composed map:*

$$(1) \quad \begin{aligned} f : M &\rightarrow Gr_{N-1}(W) \times \cdots \times Gr_{N-1}(W) \rightarrow Gr_{q(N-1)}(\mathbb{C}^n), \\ x &\mapsto (f_0(x), \dots, f_0(x)) \mapsto f_0(x) \oplus \cdots \oplus f_0(x). \end{aligned}$$

G -equivariance of a holomorphic map $f : M \rightarrow Gr_p(\mathbb{C}^n)$ means that there exists a Lie group homomorphism $\rho : G \rightarrow SU(n)$ which satisfies the following equation:

$$(2) \quad f(gx) = \rho(g)f(x), \quad \text{for } x \in M, g \in G.$$

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2. PRELIMINARIES

For a detail of the argument of this section, see [6]. Let \mathbb{C}^n be an n -dimensional complex vector space with a Hermitian inner product $(\cdot, \cdot)_n$ and $Gr_p(\mathbb{C}^n)$ be the complex Grassmannian manifold of complex p -planes in \mathbb{C}^n . We denote by $S \rightarrow Gr$ the tautological bundle and by $\underline{\mathbb{C}^n} := Gr_p(\mathbb{C}^n) \times \mathbb{C}^n \rightarrow Gr$ the trivial bundle over $Gr_p(\mathbb{C}^n)$. They are holomorphic vector bundles. The trivial bundle $\underline{\mathbb{C}^n} \rightarrow Gr$ has a Hermitian fiber metric induced by $(\cdot, \cdot)_n$, which is denoted by the same notation. Since $S \rightarrow Gr$ is a subbundle of $\underline{\mathbb{C}^n} \rightarrow Gr$, The bundle $S \rightarrow Gr$ has a Hermitian fiber metric h_S induced by $(\cdot, \cdot)_n$ and we obtain a holomorphic vector bundle $Q \rightarrow Gr$ satisfying the following short exact sequence:

$$(3) \quad 0 \rightarrow S \rightarrow \underline{\mathbb{C}^n} \rightarrow Q \rightarrow 0.$$

This is called the *universal quotient bundle* over $Gr_p(\mathbb{C}^n)$. When we denote by $S^\perp \rightarrow Gr$ the orthogonal complement bundle of $S \rightarrow Gr$ in $\underline{\mathbb{C}^n} \rightarrow Gr$, $Q \rightarrow Gr$

is isomorphic to $S^\perp \rightarrow Gr$ as a C^∞ -complex vector bundle. Thus $Q \rightarrow Gr$ has a Hermitian fiber metric h_Q induced by the Hermitian fiber metric of $S^\perp \rightarrow Gr$.

These vector bundles are all homogeneous vector bundles. We set $\tilde{G} := SU(n)$ and $\tilde{K} := S(U(p) \times U(q))$. Then $Gr_p(\mathbb{C}^n) \cong \tilde{G}/\tilde{K}$. Let \mathbb{C}^p be a p -dimensional complex subspace of \mathbb{C}^n such that \mathbb{C}^p is an irreducible representation space of \tilde{K} . We denote by \mathbb{C}^q the orthogonal complement space of \mathbb{C}^p in \mathbb{C}^n , which is also an irreducible representation space of \tilde{K} . Then $S \rightarrow Gr$ and $S^\perp \rightarrow Gr$ are expressed as the following:

$$S = \tilde{G} \times_{\tilde{K}} \mathbb{C}^p, \quad S^\perp = \tilde{G} \times_{\tilde{K}} \mathbb{C}^q.$$

For the exact sequence (3), the inclusion $S \rightarrow \underline{\mathbb{C}^n}$ is expressed as

$$S = \tilde{G} \times_{\tilde{K}} \mathbb{C}^p \ni [g, v] \mapsto ([g], gv) \in \tilde{G}/\tilde{K} \times \mathbb{C}^n = \underline{\mathbb{C}^n},$$

for $g \in \tilde{G}$ and $v \in \mathbb{C}^p$. Similarly $Q \rightarrow Gr$ is regarded as a subbundle of $\underline{\mathbb{C}^n} \rightarrow M$:

$$Q \cong S^\perp \ni [g, v] \mapsto ([g], gv) \in \underline{\mathbb{C}^n},$$

for $g \in \tilde{G}$ and $v \in \mathbb{C}^q$. When we regard $Q \rightarrow Gr$ a subbundle of $\underline{\mathbb{C}^n} \rightarrow Gr$ as above, the action of G to $Q \rightarrow Gr$ is expressed as the following:

$$(4) \quad g \cdot ([\tilde{g}], \tilde{g}v) = (g \cdot [\tilde{g}], g\tilde{g}v), \quad \text{for } g, \tilde{g} \in \tilde{G}, v \in \mathbb{C}^q.$$

Since the holomorphic tangent bundle $T_{1,0}Gr \rightarrow Gr$ over $Gr_p(\mathbb{C}^n)$ is identified with $S^* \otimes Q \rightarrow Gr$, where $S^* \rightarrow Gr$ is the dual bundle of $S \rightarrow Gr$, complex manifold $Gr_p(\mathbb{C}^n)$ has a homogeneous Hermitian metric $h_{Gr} := h_{S^*} \otimes h_Q$. This is called the Hermitian metric of Fubini-Study type of $Gr_p(\mathbb{C}^n)$ induced by $(\cdot, \cdot)_n$.

Remark 1. When we consider the case that $p = n - 1$ ¹, $(Gr_{n-1}(\mathbb{C}^n), h_{Gr})$ is the complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 2. (See [4].)

Let M be a compact complex manifold, $V \rightarrow M$ a holomorphic vector bundle with a Hermitian fiber metric h_V and W the space of holomorphic sections of $V \rightarrow M$. We denote by $(\cdot, \cdot)_W$ the L_2 -Hermitian inner product of W . Let \tilde{W} be a subspace of W and we denote by ev a bundle homomorphism:

$$ev : \underline{\tilde{W}} := M \times \tilde{W} \rightarrow V : (x, t) \mapsto t(x).$$

This is called an *evaluation map*. We assume that the evaluation map ev is surjective. In this case, $V \rightarrow M$ is called *globally generated by \tilde{W}* . For each $x \in M$ we set the linear map $ev_x : \tilde{W} \rightarrow V_x : t \mapsto t(x)$. Since $V \rightarrow M$ is globally generated by \tilde{W} , ev_x is surjective for each $x \in M$. Thus the dimension of the kernel $\text{Ker} ev_x$ of ev_x are independent of x , which is denoted by p . Therefore we obtain a holomorphic map

$$f : M \rightarrow Gr_p(\tilde{W}) : x \mapsto \text{Ker} ev_x.$$

This is called a *induced map* by $(L \rightarrow M, \tilde{W})$. When $\tilde{W} = W$, the induced map is called *standard map* by $L \rightarrow M$.

On the other hand, let $f : M \rightarrow Gr_p(\mathbb{C}^n)$ be a holomorphic map and $f^*Q \rightarrow M$ the pull-back vector bundle of $Q \rightarrow Gr$ by f with pull-back metric h_Q and connection ∇^Q . We assume that $f^*Q \rightarrow M$ is isomorphic to $V \rightarrow M$ as a holomorphic

¹In this paper, the complex projective space means the complex Grassmannian manifold $Gr_{n-1}(\mathbb{C}^n)$, not $Gr_1(\mathbb{C}^n)$.

Hermitian vector bundle. For [6] there exists a semi-positive Hermitian endomorphism T of W such that f is expressed as the following map:

$$(5) \quad f : M \longrightarrow Gr_p(W) : x \longmapsto T^{-1}(f_0(x) \cap (\text{Ker } T)^\perp),$$

where T^{-1} is the inverse of $T : \text{Ker } T^\perp \longrightarrow \text{Ker } T^\perp$. The semi-positive Hermitian endomorphism $T : W \longrightarrow W$ is obtained by the following construction: by Borel-Weil Theory the complex vector space \mathbb{C}^n is regarded as the space of holomorphic sections of $Q \rightarrow Gr_p(\mathbb{C}^n)$. We have a linear map $\iota : \mathbb{C}^n \longrightarrow W$ by restricting sections to M .

Definition 2 ([6]). A holomorphic map $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is called *full* if $\iota : \mathbb{C}^n \longrightarrow W$ is injective.

We assume that $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is full. Then \mathbb{C}^n can be considered as a subspace of W by ι . Let $ev_{\mathbb{C}} : \underline{\mathbb{C}^n} \longrightarrow V$ and $ev : \underline{W} \longrightarrow V$ be evaluation maps. Then for any $x \in M$, $\text{Ker } ev_{\mathbb{C}_x} = \text{Ker } ev_x \cap \mathbb{C}^n$. Therefore $f : M \longrightarrow (Gr_p(\mathbb{C}^n), (\cdot, \cdot)_n)$ is expressed as

$$f(x) = \text{Ker } ev_x \cap \mathbb{C}^n.$$

The Hermitian inner product $(\cdot, \cdot)_n$ is not always coincide with $(\cdot, \cdot)_W$. Let \underline{T} be the positive Hermitian endomorphism of \mathbb{C}^n which satisfies that

$$(6) \quad (\underline{T}u, \underline{T}v)_n = (u, v)_W$$

for any $u, v \in \mathbb{C}^n$. We have an isometry

$$\underline{T}^{-1} : (Gr_p(\mathbb{C}^n), (\cdot, \cdot)_n) \longrightarrow (Gr_p(\mathbb{C}^n), (\cdot, \cdot)_W) : U \longmapsto \underline{T}^{-1}U.$$

Let $\pi : W \longrightarrow \mathbb{C}^n$ be the orthogonal projection onto \mathbb{C}^n with respect to $(\cdot, \cdot)_W$. We denote by $T := \underline{T} \circ \pi$ an endomorphism of W , which is semi-positive Hermitian. Consequently $f : M \longrightarrow Gr_p(\mathbb{C}^n)$ is expressed as

$$(7) \quad f : M \longrightarrow (Gr_p(W), (\cdot, \cdot)_W) : x \longmapsto T^{-1}(f_0(x) \cap \text{Ker } T^\perp).$$

This means that a holomorphic map which has the pull-back bundle $f^*Q \rightarrow M$ isomorphic to $V \rightarrow M$ is expressed a deformation of the standard map induced by $V \rightarrow M$.

Let $M = G/K$ be a compact homogeneous Kähler manifold, V_0 a irreducible K -representation space and $V = G \times_K V_0 \rightarrow M$ a holomorphic homogeneous vector bundle. We denote by W the space of holomorphic sections of V with L_2 -Hermitian inner product $(\cdot, \cdot)_W$. It follows from Bott-Borel-Weil theory that W is an irreducible G -representation space. For $g \in G$ and $t \in W$, the action of G to W is expressed as

$$(g \cdot t)([g_0]) := g(t(g^{-1}[g_0])), \quad g_0 \in G.$$

We assume that the evaluation map $ev : M \times W \longrightarrow V$ is surjective.

Proposition 2 ([6]). V_0 is considered as a subspace of W .

We set $\pi_0 := ev_{[e]} : W \longrightarrow V_0$. For $g \in G$ and $t \in W$, we can calculate

$$(8) \quad \begin{aligned} ev([g], t) &= t([g]) = g \cdot g^{-1}t(g \cdot [e]) \\ &= g \cdot ev([e], g^{-1}t) = g[e, \pi_0(g^{-1}t)] = [g, \pi_0(g^{-1}t)]. \end{aligned}$$

The adjoint map ev^* of ev is expressed as $ev^*([g, v]) = ([g], gv)$.

Let $U_0 := \text{Ker} ev_{[e]}$. Then U_0 is also K -representation space and W is orthogonal direct sum of U_0 and V_0 . It follows from (8) that $\text{Ker} ev_{[g]} = gU_0$. Therefore the standard map $f_0 : M \rightarrow Gr(W)$ induced by $V \rightarrow M$ is that

$$(9) \quad f_0([g]) = gU_0.$$

Thus the standard map induced by a holomorphic homogeneous vector bundle is G -equivariant.

Let $f : M \rightarrow Gr_p(\mathbb{C}^n)$ be a full holomorphic map such that $f^*Q \rightarrow M$ is holomorphic isomorphic to $V \rightarrow M$. Then there exists a semi-positive Hermitian endomorphism $T : W \rightarrow W$ such that the map f is expressed as the following:

$$f : M \rightarrow (Gr_p(\mathbb{C}^n), (\cdot, \cdot)_W) : [g] \mapsto T^{-1}(gU_0 \cap (\text{Ker} T)^\perp).$$

Pulling the sequence (3) back, we have a short exact sequence:

$$(10) \quad 0 \rightarrow f^*S \rightarrow \underline{\mathbb{C}^n} := M \times \mathbb{C}^n \rightarrow f^*Q \rightarrow 0.$$

The vector bundle $f^*Q \rightarrow M$ is regarded as a subbundle of $\underline{\mathbb{C}^n} \rightarrow M$, which is orthogonal complement bundle of $f^*S \rightarrow M$:

$$f^*Q = \{([g], v) \in \underline{\mathbb{C}^n} | f([g]) \perp v\}.$$

It follows from (5) that we have

$$0 = (v, T^{-1}(gU_0 \cap \mathbb{C}^n))_W = (g^{-1}T^{-1}v, U_0 \cap \mathbb{C}^n)_W.$$

Then we have $g^{-1}T^{-1}v \in V_0 \iff v \in TgV_0$. Therefore the isomorphism $\phi : V \rightarrow f^*Q \subset \underline{\mathbb{C}^n}$ is expressed as the following:

$$(11) \quad [g, v] \mapsto ([g], Tgv), \quad \text{for } g \in G, v \in V_0.$$

3. MAIN THEOREM AND ITS PROOF

Let M be a compact simply connected homogeneous Kähler manifold, G the isometry group of M and K an isotropy subgroup of G . Let $f : M \rightarrow Gr_p(\mathbb{C}^n)$ be a full holomorphic strongly projectively flat map. By definition of strongly projectively flatness there exists a Hermitian line bundle $L \rightarrow M$ such that $f^*Q \rightarrow M$ is isomorphic to $\tilde{L} \rightarrow M$ as a Hermitian vector bundle, where $\tilde{L} \rightarrow M$ is orthogonal direct sum of q -copies of $L \rightarrow M$. Since M is a compact simply connected homogeneous Kähler, $L \rightarrow M$ is homogeneous. We set L_0 the 1-dimensional K -representation space such that $L = G \times_K L_0$. Then we have

$$\tilde{L} = L \oplus \cdots \oplus L = G \times_K (L_0 \oplus \cdots \oplus L_0) = G \times_K \tilde{L}_0,$$

where \tilde{L}_0 is q -orthogonal direct sum of L_0 . We denote by W and \tilde{W} the spaces of holomorphic sections of $L \rightarrow M$ and $\tilde{L} \rightarrow M$ respectively and we set N the dimension of W . By definition of $\tilde{L} \rightarrow M$, \tilde{W} is regarded as q -orthogonal direct sum of W . Let $\pi_j : \tilde{W} \rightarrow W$ be the orthogonal projection onto the j -th component of \tilde{W} . It follows from Proposition 2 that L_0 is a subspace of W and \tilde{L}_0 is a subspace of \tilde{W} as a K -representation space. When we restrict π_j to \tilde{L}_0 , $\pi_j|_{\tilde{L}_0}$ is orthogonal projection of \tilde{L}_0 onto the j -th component of \tilde{L}_0 . We denote by

$$ev : M \times W \rightarrow L, \quad \tilde{ev} : M \times \tilde{W} \rightarrow \tilde{L}$$

the evaluation maps respectively and by $f_0 : M \rightarrow Gr_{N-1}(W)$ and $\tilde{f}_0 : M \rightarrow Gr_{q(N-1)}(\tilde{W})$ the standard maps induced by $L \rightarrow M$ and $\tilde{L} \rightarrow M$ respectively. Since \tilde{W} is orthogonal direct sum of q -copies of W , we have

$$(12) \quad \begin{aligned} \tilde{e}v([g], t) &= \tilde{e}v([g], t_1 \oplus \cdots \oplus t_q) \\ &= ev([g], t_1) \oplus \cdots \oplus ev([g], t_q) \in L \oplus \cdots \oplus L, \end{aligned}$$

where $t = t_1 \oplus \cdots \oplus t_q$ is orthogonal decomposition with respect to $\tilde{W} = W \oplus \cdots \oplus W$. We set $U_0 := \text{Ker} ev|_{[e]}$. Then it follows from (9) that $f_0([g]) = gU_0$. It follows from (12) that the map \tilde{f}_0 is expressed as

$$(13) \quad \begin{aligned} \tilde{f}_0 : M &\rightarrow Gr_{N-1}(W) \times \cdots \times Gr_{N-1}(W) \rightarrow Gr_{q(N-1)}(\tilde{W}), \\ [g] &\mapsto (gU_0, \dots, gU_0) \mapsto gU_0 \oplus \cdots \oplus gU_0 = g \cdot (U_0 \oplus \cdots \oplus U_0). \end{aligned}$$

Since $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is full, \mathbb{C}^n is a subspace of \tilde{W} . It follows from the previous section that there exists a semi-positive Hermitian endomorphism T of \tilde{W} and a bundle isomorphism $\phi : \tilde{L} \rightarrow f^*Q$ such that maps $f : M \rightarrow Gr_p(\mathbb{C}^n)$ and $\phi : \tilde{L} \rightarrow f^*Q$ is expressed as the following:

$$(14) \quad f([g]) = T^{-1} \left(\tilde{f}_0([g]) \cap (\text{Ker} T)^\perp \right),$$

$$(15) \quad \phi([g], v) = ([g], Tgv),$$

for $g \in G$ and $v \in \tilde{L}_0$.

By using the above notations, we rewrite Main theorem as followings:

Theorem 2. *Let M be a compact simply connected homogeneous Kähler manifold and G the isometry group of K . Let $f : M \rightarrow Gr_p(\mathbb{C}^n)$ be a full holomorphic strongly projectively flat map into the complex Grassmannian manifold. Then f is G -equivariant if and only if f is the standard map.*

In order to prove this theorem, it is sufficient to show that the Hermitian endomorphism $T : \tilde{W} \rightarrow \tilde{W}$ is the identity map of \tilde{W} .

From now on, we assume that $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is G -equivariant. Then there exists a Lie group homomorphism $\rho : G \rightarrow SU(n)$ which satisfy the following equation:

$$(16) \quad f(g[\tilde{g}]) = \rho(g)f([\tilde{g}]), \quad g, \tilde{g} \in G.$$

By definition \mathbb{C}^n is G -representation space and a vector subspace of \tilde{W} .

Lemma 1. *$f^*Q \rightarrow M$ is homogeneous.*

Proof. The definition of the pull-back bundle $f^*Q \rightarrow M$ is that

$$f^*Q = \{([g], v) \in M \times Q \mid f([g]) = \pi(v)\},$$

where $\pi : Q \rightarrow Gr_p(\mathbb{C}^n)$ is the natural projection. For any $([\tilde{g}], v) \in f^*Q$ and $g \in G$, we have an action of G to $f^*Q \rightarrow M$ by

$$g \cdot ([\tilde{g}], v) = (g[\tilde{g}], \rho(g)v).$$

Since G acts to M transitively, $f^*Q \rightarrow M$ is homogeneous. \square

Since $f^*Q \rightarrow M$ is homogeneous, the space of holomorphic sections of $f^*Q \rightarrow M$ is G -representation space. Let t be a holomorphic section of $f^*Q \rightarrow M$. For $g \in G$ and $x \in M$, we have

$$(g \cdot t)(x) = g(t(g^{-1}x)).$$

For $t \in \mathbb{C}^n$, we obtain a holomorphic section of $f^*Q \rightarrow M$ which is expressed as

$$t(x) = (x, t(f(x))), \quad \text{for } x \in M.$$

Thus for $g, \tilde{g} \in G$ we obtain

$$\begin{aligned} (g \cdot t)(x) &= g(t(g^{-1}x)) = g(g^{-1}x, t(f(g^{-1}x))), \\ &= (x, \rho(g)t(\rho(g^{-1})f(x))) = (x, (\rho(g)t)(f(x))), \\ &= (\rho(g)t)(x). \end{aligned}$$

Therefore \mathbb{C}^n is a G -representation subspace of the space of holomorphic sections of $f^*Q \rightarrow M$.

Lemma 2. *The holomorphic isomorphism $\phi : \tilde{L} \rightarrow f^*Q$ is G -equivariant.*

Proof. At first we show that $f^*Q \rightarrow M$ is isomorphic to \tilde{L} as a homogeneous vector bundle. Since ϕ preserves Hermitian connections, bundles $\tilde{L} \rightarrow M$ and $f^*Q \rightarrow M$ have same holonomy groups and ϕ is holonomy equivariant. Since the action of K to $f^*Q \rightarrow M$ and $L \rightarrow M$ at $[e]$, where e is the unit element in G , is expressed as a action of the holonomy group, ϕ is K -equivariant. Thus $f^*Q_{[e]}$ is isomorphic to $L_0 \oplus \cdots \oplus L_0$ as a K -representation space. Therefore $f^*Q \rightarrow M$ is isomorphic to $\tilde{L} \rightarrow M$ as a homogeneous vector bundle.

Finally we show that a holomorphic isomorphism $\phi : \tilde{L} \rightarrow \tilde{L}$ is G -equivariant. We denote by $\tilde{L} \cong L_1 \oplus \cdots \oplus L_q$ and $L_j = G \times_K L_{(j)}$, where $L_j \rightarrow M$ is the j -th component and $L_{(j)}$ is isomorphic to L_0 as a K -representation space for $j = 1, \dots, q$. Then we have

$$\tilde{L} = G \times_K (L_{(1)} \oplus \cdots \oplus L_{(q)}).$$

Let $\phi_j : L_j \rightarrow \tilde{L}$ be the restriction of $\phi : \tilde{L} \rightarrow \tilde{L}$ to $L_j \rightarrow M$. Then ϕ_j is expressed as the following:

$$\phi_j([g, v]) = [g, \varphi_1(g)(v) \oplus \cdots \oplus \varphi_q(g)(v)], \quad \text{for } g \in G, v \in L_{(j)},$$

where $\varphi_i(g) : L_{(j)} \rightarrow L_{(i)}$ is a linear map for $i = 1, \dots, q$. Since $L_{(i)}$ and $L_{(j)}$ are isomorphic 1-dimensional K -representation spaces, there exists a complex number $\alpha_i(g)$ such that $\varphi_i(g)(v) = \alpha_i(g)v$. Thus we have

$$(17) \quad \phi_j([g, v]) = [g, \alpha_1(g)v \oplus \cdots \oplus \alpha_q(g)v], \quad \text{for } g \in G, v \in L_{(j)}.$$

Since ϕ_j is a bundle homomorphism, we obtain

$$\begin{aligned} \phi_j([gk, v]) &= [gk, \alpha_1(gk)v \oplus \cdots \oplus \alpha_q(gk)v] = [g, \alpha_1(gk)kv \oplus \cdots \oplus \alpha_q(gk)kv], \\ \phi_j([g, kv]) &= [g, \alpha_1(g)kv \oplus \cdots \oplus \alpha_q(g)kv], \end{aligned}$$

for $g \in G, k \in K$ and $v \in L_{(j)}$. It follows that $\alpha_i(gk) = \alpha_i(g)$ for $i = 1, \dots, q, g \in G$ and $k \in K$. Therefore α_i is a complex valued function on G/K . Since ϕ_j is holomorphic, so is α_i for $i = 1, \dots, q$, which implies that α_i is a constant function for each i because G/K is compact. We regard α_i as a complex number. Then we have

$$\phi_j([g, v]) = [g, \alpha_1 v \oplus \cdots \oplus \alpha_q v], \quad \text{for } g \in G, v \in L_{(j)}.$$

This is G -equivariant for each $j = 1, \dots, q$. Consequently $\phi : \tilde{L} \rightarrow \tilde{L}$ is G -equivariant. \square

It follows from Lemma 1 and Lemma 2 that \mathbb{C}^n is a G -representation subspace of \mathbb{C}^n .

Lemma 3. *The semi-positive Hermitian endomorphism $T : \tilde{W} \rightarrow \tilde{W}$ is G -equivariant.*

Proof. Since $\phi : \tilde{L} \rightarrow f^*Q$ is G -equivariant, it follows from (4) and (15) that we can calculate

$$\begin{aligned}\phi(g_1 \cdot [g_2, v]) &= \phi([g_1 g_2, v]) = ([g_1 g_2], T g_1 g_2 v), \\ \phi(g_1 \cdot [g_2, v]) &= g_1 \cdot \phi([g_2, v]) = g_1 \cdot ([g_2], T g_2 v) = ([g_1 g_2], g_1 T g_2 v).\end{aligned}$$

Therefore we have

$$T g v = g T v, \quad \text{for } g \in G, v \in \tilde{L}_0.$$

We denote by GL_0 an subspace of W spanned by gv for any $g \in G$ and $v \in L_0$ and similarly we denote by $G\tilde{L}_0$. Then $G\tilde{L}_0$ is regarded as q -orthogonal direct sum of GL_0 . GL_0 is a G -representation subspace of W . Since W is irreducible and GL_0 is not empty, we obtain $W = GL_0$. Consequently we obtain $\tilde{W} = G\tilde{L}_0$. It follows that for any $w \in \tilde{W}$ there exists $\alpha_i \in \mathbb{C}$, $g_i \in G$ and $v_i \in \tilde{L}_0$ such that $w = \sum \alpha_i g_i v_i$, where the right hand side of this equation is a finite sum. For any $g \in G$, we have

$$T g w = T g \sum \alpha_i g_i v_i = \sum T g \alpha_i g_i v_i = \sum g T \alpha_i g_i v_i = g T \sum \alpha_i g_i v_i = g T w.$$

Therefore T is G -equivariant. \square

Since $T : \tilde{W} \rightarrow \tilde{W}$ is G -equivariant, T is also K -equivariant.

Lemma 4.

$$T(\tilde{L}_0) \subset \tilde{L}_0.$$

Proof. Since the orthogonal projection $\pi_j : \tilde{W} \rightarrow W$ is K -equivariant for each $j = 1, \dots, q$, $\pi_j \circ T : \tilde{W} \rightarrow W$ is a K -equivariant endomorphism. Thus $\pi_j \circ T(\tilde{L}_0) \subset W$ is a K -representation subspace of W . It follows from Schur's lemma and Borel-Weil theory that $\pi_j \circ T(\tilde{L}_0) \subset L_0$. Consequently $T(\tilde{L}_0) \subset (\tilde{L}_0)$. \square

We denote by the same notation $T : \tilde{L}_0 \rightarrow \tilde{L}_0$ the restriction of $T : \tilde{W} \rightarrow \tilde{W}$ to \tilde{L}_0 .

Theorem 3. *The endomorphism $T : \tilde{W} \rightarrow \tilde{W}$ is the identity map.*

Proof. Since the bundle isomorphism $\phi : \tilde{L} \rightarrow f^*Q$ preserves fiber metrics and T is Hermitian, we have

$$(v_1, v_2)_{\tilde{L}_0} = ([e, v_1], [e, v_2])_{\tilde{L}} = ([e, T v_1], [e, T v_2])_{\tilde{L}} = (T v_1, T v_2)_{\tilde{L}_0} = (T^2 v_1, v_2)_{\tilde{L}_0},$$

for any $v_1, v_2 \in \tilde{L}_0$. Therefore $T^2 : \tilde{L}_0 \rightarrow \tilde{L}_0$ is the identity map. Since W is G -irreducible and T is G -equivariant, $T^2 : \tilde{W} \rightarrow \tilde{W}$ is the identity map and so is T because T is semi-positive Hermitian. \square

Consequently, a holomorphic strongly projectively flat G -equivariant map $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is the standard map induced by a q -orthogonal direct sum bundle of Hermitian line bundle $L \rightarrow M$.

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GRADUATE SCHOOL OF MATHEMATICS,, KYUSHU UNIVERSITY,, 744 MOTOOKA, NISHI-KU, FUKUOKA, 819-0395, JAPAN.

E-mail address: i-koga@math.kyushu-u.ac.jp